Calculating the two-photon contribution to the real part of $\pi^0 \to e^+e^-$ decay amplitude

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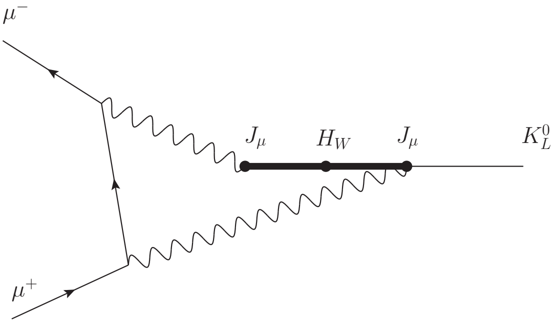
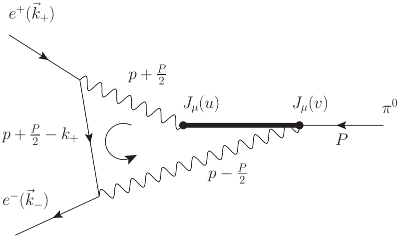
Two-photon contribution to $K_L \to \mu^+\mu^-$



- Intermediate states with lower energy than kaon mass: $\gamma\gamma, |0\rangle, \eta, \pi, \pi\pi(\gamma)$, etc.
- Calculation in Euclidean space would result in exponentially divergent behavior.

$\pi^0 \to e^+e^-$



$$\langle e^+(k_+)e^-(k_-)|\pi^0(P)\rangle = \int d^4u\ d^4v\ H_{\mu\nu}(u,v)L^{\mu\nu}(u,v)$$

$$H_{\mu\nu}(u,v) = \langle 0|J_\mu(u)J_\nu(v)|\pi\rangle$$

$$L^{\mu\nu}(u,v) = \int d^4p\ e^{-ip\cdot(u-v)} \left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 - i\epsilon}\right] \left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 - i\epsilon}\right]$$

$$\overline{u}(k_-)\gamma_{\mu'} \left[\frac{\gamma\cdot(p+\frac{P}{2}-k_-)+m_e}{(p+\frac{P}{2}-k_-)^2+m_e^2-i\epsilon}\right] \gamma_{\nu'} v(k_+)$$

$\pi^0 \to e^+e^-$

Let w be the relative coordinate of two EM currents i.e. $w = u - v$. Decay amplitude:

$$\begin{aligned}
\mathcal{A} \;&=\; \int d^4w\, \langle 0|\, T\{J_\mu(\frac{w}{2}) J_\nu(-\frac{w}{2})\}|\pi^0\rangle \qquad (1)\\
&\int d^4p\; e^{-ip\cdot w}\left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 + m_\gamma^2 - i\epsilon}\right]\left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 + m_\gamma^2 - i\epsilon}\right]\\
&\qquad \overline{u}(k_-)\gamma_{\mu'}\left[\frac{\gamma\cdot(p+\frac{P}{2}-k_-)+m_e}{(p+\frac{P}{2}-k_-)^2+m_e^2-i\epsilon}\right]\gamma_{\nu'}v(k_+)
\end{aligned}$$

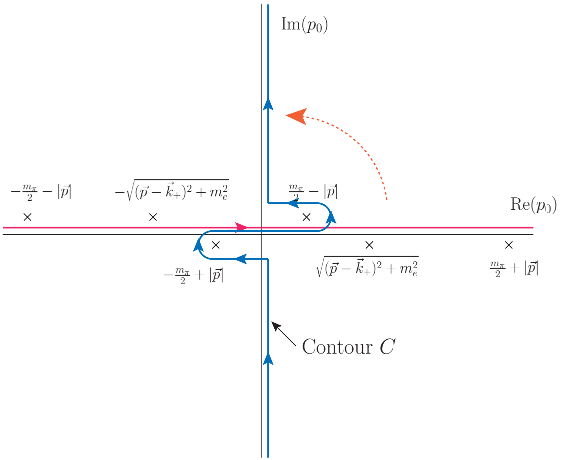
Here the matrix elment is a Minkowski-space quantity.
To calculate it using Lattice QCD, it is necessary to do a Wick rotation $w^0 \to iw^0$.

Wick Rotation

$$
\begin{aligned}
L_{\mu\nu}(w) \;\; = \;\; & \int d^4p \; e^{-ip\cdot w} \left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 - i\epsilon} \right] \left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 - i\epsilon} \right] \\
& \overline{u}(k_-)\gamma_{\mu'} \left[\frac{\gamma\cdot(p+\frac{P}{2}-k_-) + m_e}{(p+\frac{P}{2}-k_-)^2 + m_e^2 - i\epsilon} \right] \gamma_{\nu'} v(k_+)
\end{aligned}
$$

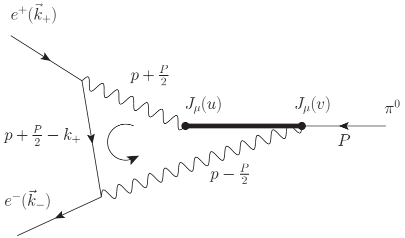
- After w^0 is Wick-rotated, the exponential in $L_{\mu\nu}(w)$ introduces exponential growth.
- The p^0 contour also needs to be rotated.
- Because of the presence of intermediate states with lower energy than the mass of pion, a naive Wick rotation $p^0 \to ip_E^0$ will make the integral blow up.

Wick Rotation



- When $|\vec{p}| < \frac{m_\pi}{2}$, the p^0 contour needs to be deformed to circumvent the two poles that crosses the imaginary axis.

Wick Rotation

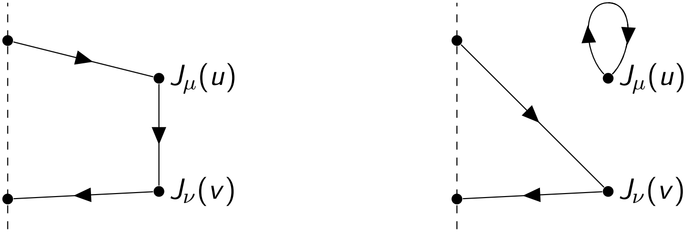


- $L_{\mu\nu} \propto e^{M_\pi \frac{|w^0|}{2}}$, where $w^0 = u^0 - v^0$
- The lightest intermediate state between J_μ and J_ν is $\pi\pi$ state $\rightarrow H_{\mu\nu} = \langle 0|J_\mu(\frac{w}{2})J_\nu(-\frac{w}{2})|\pi\rangle \propto e^{-(E_n - \frac{M_\pi}{2})|w^0|}$, $E_n > 2M_\pi$
- Amplitude converges exponentially at least as fast as $e^{-M_\pi|w^0|}$. We can safely perform the Euclidean space calculation.

Hadronic part

$$\langle 0|TJ_\mu(0)J_\nu(x)|\pi\rangle = \lim_{t\to-\infty} \frac{2m_\pi}{N_\pi} Z_V^2 e^{m_\pi|t|}\langle 0|TJ_\mu(0)J_\nu(x)\pi(t)|0\rangle$$

where N_π is the pion ground state amplitude $N_\pi = \langle\pi|\pi(0)|0\rangle$, Z_V is the coefficient of EM current that connects the local non-conserved current with global conserved current.



(a) Connected Diagram (b) Disconnected Diagram

- Contribution from the second graph is small (suppressed by SU(3) flavor symmetry).

Summary

- Leptonic part integral is calculated numerically with CUBA library.
- Hadronic part can be extracted from three point function.

$$\langle 0|TJ_\mu(0)J_\nu(x)|\pi\rangle = \lim_{t\to-\infty}\frac{2m_\pi}{N_\pi}Z_V^2 e^{m_\pi|t|}\langle 0|TJ_\mu(0)J_\nu(x)\pi(t)|0\rangle$$

What we have calculated:

- Imaginary part of the decay amplitude
  It can be compared with Optical theorem prediction to help us estimate error in lattice calculation.
- Real part of the decay amplitude (Our real goal)

Ensembles

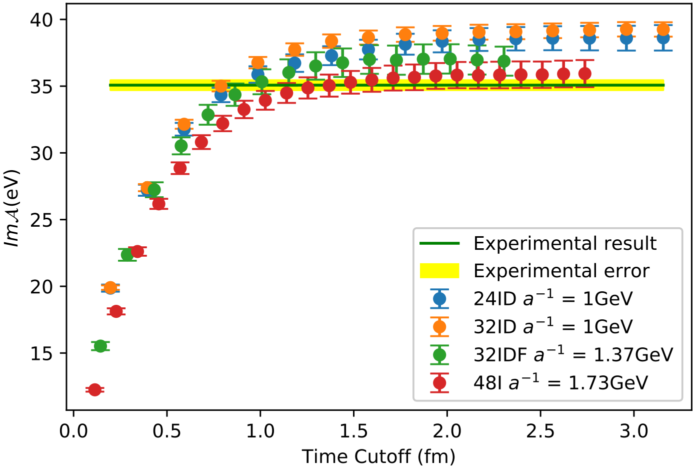
1. $24^3 \times 64$ **Iwasaki + DSDR**
Lattice volume: $24^3 \times 64$
$a^{-1} = 1 GeV$
$am_\pi = 0.13975(10)$
Number of trajectories: 35

2. $32^3 \times 64$ **Iwasaki + DSDR**
Lattice volume: $32^3 \times 64$
$a^{-1} = 1 GeV$
$am_\pi = 0.139474(96)$
Number of trajectories: 40

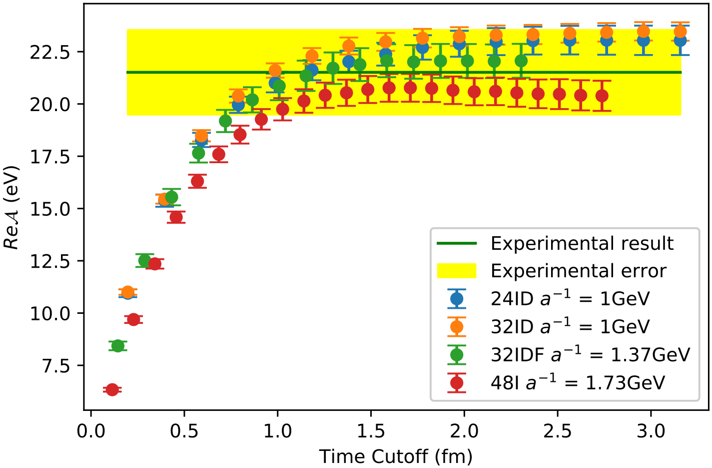
3. $32^3 \times 64$ **Iwasaki + DSDR**
Lattice volume: $32^3 \times 64$
$a^{-1} = 1.37 GeV$
$am_\pi = 0.10468(32)$
Number of trajectories: 17

4. $48^3 \times 96$ **Iwasaki**
Lattice volume: $32^3 \times 64$
$a^{-1} = 1.73 GeV$
$am_\pi = 0.08049(13)$
Number of trajectories: 20

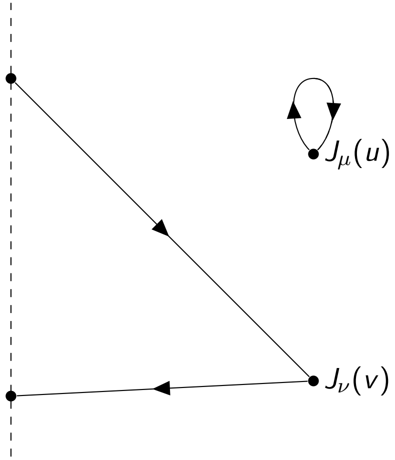
Imaginary Part Contribution to Decay Rate



Real Part Contribution to Decay Rate



Disconnected Diagram



Disconnected Diagram

- EM current tadpole $Tr[\gamma_\mu S(x,x)]$ is calculated using random grid source.
- Disconnected diagram is about 2% of the connected diagram and they have the opposite sign.
- Disconnected diagram is treated as a part of systematic error.

Conclusion

Experimental result from PDG:

$$\mathrm{Im}\mathcal{A} = 35.07(37) \text{ eV} \tag{2}$$
$$\mathrm{Re}\mathcal{A} = 21.51(2.02) \text{ eV} \tag{3}$$

Dispersion relation [Weil et al., 2017]:

$$\mathrm{Re}\mathcal{A} = 20.16(23) \text{ eV}$$

The error mostly comes from the experimental pion life time.

The result we obatined from lattice calculation:

$$\mathrm{Im}\mathcal{A} = 35.94(1.01)(1.19) \text{ eV} \tag{4}$$
$$\mathrm{Re}\mathcal{A} = 20.39(72)(70) \text{ eV} \tag{5}$$

Conclusion

- We developed a method for dealing with two-photon intermediate state and combining QED and QCD part of the amplitude
- Using this method, we carried out a first-principles calculation of $\pi^0 \to e^+e^-$ decay
- The next step is to calculate a more interesting process – the two photon contribution to $K_L \to \mu^+\mu^-$ decay